Asymptotics of the GLRT for the disorder problem in diffusion processes

Fabien Campillo, François Le Gland, Yurii Kutoyants

UNITÉ DE RECHERCHE
INRIA-SOPHIA ANTIPOLIS

Institut National
de Recherche
en Informatique
et en Automatique

Sophia Antipolis
B.P. 109
06561 Valbonne Cedex
France
Tél.: 93 65 77 77

Rapports de Recherche

N°1735

Programme 5
Traitement du Signal,
Automatique et Productique

ASYMPTOTICS OF THE GLRT FOR THE DISORDER PROBLEM IN DIFFUSION PROCESSES

Fabien CAMPILLO
François LE GLAND
Yurii KUTOYANTS

Juillet 1992

ASYMPTOTICS OF THE GLRT FOR THE DISORDER PROBLEM IN DIFFUSION PROCESSES

Comportement asymptotique du test du rapport de vraisemblance généralisé pour la détection de changement dans les diffusions.

Fabien CAMPILLO
François LE GLAND
INRIA Sophia–Antipolis
BP 93
06902 SOPHIA–ANTIPOLIS Cédex
France

Yurii KUTOYANTS*
Université du Maine
Département de Mathématiques
BP 535
72017 LE MANS Cédex
France

*also Université de Paris VI, LSTA, 4 place Jussieu, 75252 PARIS Cédex 05, France

Résumé

Nous considérons le problème de détecter un changement dans le coefficient de dérive d'un processus de diffusion, en utilisant le test du rapport de vraisemblance généralisé (GLRT). Sous l'hypothèse qu'un changement a eu lieu, le comportement asymptotique (consistance, distribution asymptotique) de l'estimateur du maximum de vraisemblance pour l'instant de changement, a déja été étudié dans l'asymptotique petit bruit. Le but de cet article est d'etudier le comportement asymptotique du GLRT, c'est–à–dire le comportement des probabilités de fausse alarme et de non–détection. Nous montrons que ces probabilités convergent vers zero avec une vitesse exponentielle, pourvu qu'une hypothèse de détectabilité soit vérifiée par le système déterministe limite.

Nous nous intéressons aussi à la robustesse du GLRT vis–à–vis de possibles erreurs de modélisation, qui est une propriété très importante pour l'implémentation pratique du test. Les erreurs de modélisation peuvent porter aussi bien sur le coefficient de dérive (avant le changement), que sur le coefficient de changement. Nous obtenons le même type de comportement asymptotique que dans le cas où le modèle est correct, sous l'hypothèse supplémentaire que le changement à détecter soit plus grand, dans un certain sens, que l'erreur de modélisation.

Abstract

We consider the problem of detecting a change in the drift coefficient of a diffusion type process (disorder problem), using the generalized likelihood ratio test (GLRT). Assuming that a change has occured, the asymptotic behaviour (consistency, asymptotic probability distribution) of the maximum likelihood estimate of the change time has already been investigated in the small noise asymptotics. The purpose of this paper is to study the asymptotics of the GLRT itself, i.e. the asymptotic behaviour of both the probability of false alarm and the probability of miss detection. We prove that these probabilities go to zero with exponential rate, provided a simple detectability assumption is satisfied by the limiting deterministic system.

We also investigate the robustness of the GLRT with respect to model misspecification, which is a very important property for practical implementation. Here, misspecification means that some wrong expressions are used for either the drift coefficient (before change), or the change coefficient. We obtain roughly the same behaviour for the error probabilities, as in the correctly specified case, although the detectability assumption will include the requirement that the change to be detected is larger in some sense than the misspecification error.

Key words : stochastic differential equations, disorder, change detection, generalized likelihood ratio test, small noise asymptotics, robustness.

Contents

1 Introduction **1**

2 Statistical model **3**

2.1 Probability of false alarm 5

2.2 Probability of miss detection 6

2.3 Conclusion 9

3 Robustness w.r.t. misspecificied change **10**

3.1 Probability of false alarm 10

3.2 Probability of miss detection 10

3.3 Conclusion 12

4 Robustness w.r.t. misspecified model **14**

4.1 Probability of false alarm 14

4.2 Probability of miss detection 16

4.3 Conclusion 18

List of Figures

1 Consistency assumption for misspecified change under (H_1) 13

2 Consistency assumption for misspecified model under (H_0) 19

3 Consistency assumption for misspecified model under (H_1) 20

1 Introduction

This paper is devoted to the problem of disorder (change) detection in a diffusion type process.

Disorder refers to the switching of the drift coefficient at some unknown change time. Two types of problems can be considered. One is the detection of disorder (hypotheses testing), or quickest detection of disorder (sequential hypotheses testing). The second is the estimation of the change time (parameter estimation). Applications can be found in Basseville and Benveniste [1].

The quickest detection problem is considered in Shiryayev [6] for observations of the form

$$dX_t = a\, 1_{(t \geq \theta)}\, dt + \sigma\, dW_t \;, \quad X_0 = 0 \;, \quad t \geq 0 \;,$$

where θ is an exponential random variable on $[0, \infty)$, and the parameters a and σ are known. In this Bayesian framework, an optimal stopping time τ^* is found, which minimizes a risk function. The decision is then to accept the hypothesis that a change has occured in the time interval $[0, \tau^*]$. There has been some generalizations of this result, see for example Vostrikova [7].

The problem of estimating the change time θ is considered in Ibragimov and Khasminskii [2] for observations of the form

$$dX_t = S(t - \theta)\, dt + \varepsilon\, dW_t \;, \quad X_0 = 0 \;, \quad 0 \leq t \leq T \;,$$

where $S(\cdot)$ is some known deterministic function, with discontinuity at the origin. The asymptotic properties of the maximum likelihood and Bayes estimates of θ are described as $\varepsilon \downarrow 0$. This result was generalized to diffusion processes in Kutoyants [3].

In this paper we consider the following problem of hypotheses testing

- Under (H_0), the observations satisfy the following stochastic differential equation

$$dX_t = b_t(X)\, dt + \varepsilon\, dW_t \;, \quad X_0 = x_0 \;, \quad 0 \leq t \leq T \;,$$

- Under (H_1), there exists a change time $0 \leq \tau \leq T'$ with $T' < T$, such that

$$dX_t = [b_t(X)\, dt + a_t(X)\, 1_{(t \geq \tau)}]\, dt + \varepsilon\, dW_t \;, \quad X_0 = x_0 \;, \quad 0 \leq t \leq T \;.$$

In Section 2, we first introduce the generalized likelihood ratio test (GLRT) and we establish the exponential convergence to zero of the error probabilities, i.e. the probability of false alarm and the probability of miss detection, as $\varepsilon \downarrow 0$, provided some detectability assumption is satisfied by the limiting deterministic system.

We then investigate the robustness of the GLRT with respect to model misspecification, which is a very important property for practical implementation. This notion was introduced by McKeague in [5], for maximum likelihood parameter estimation in diffusion processes. Here, misspecification means that some wrong expressions are used for either the drift or the change coefficient. The following situations are considered, with increasing complexity

· in Section 3, only the change coefficient is misspecified,

· in Section 4, both the drift and the change coefficients are misspecified.

Under some additional conditions expressing that the modeling errors are smaller in some sense than the change to be detected, we obtain roughly the same exponential convergence to zero of the error probabilities, as $\varepsilon \downarrow 0$. In addition, the true change time can be estimated exactly under the alternate hypothesis (H_1), even though the model used is wrong. This again is an interesting feature of the GLRT.

2 Statistical model

Let $\{X_t\,,\,0 \leq t \leq T\}$ denote the canonical process on the space $\Omega = C([0,T]\,;\,\mathbf{R}^m)$, and consider the problem of deciding between the following two hypotheses

- Under the null hypothesis (H_0), there exists a probability measure P^ε such that

$$dX_t = b_t(X)\,dt + \varepsilon\,dW_t^\varepsilon\;, \quad X_0 = x_0\;, \quad 0 \leq t \leq T\;, \tag{1}$$

  where $\{W_t^\varepsilon\,,\,0 \leq t \leq T\}$ is a m–dimensional Wiener process under P^ε.

- Under the alternate hypothesis (H_1), there exists a change time $0 \leq \tau \leq T'$ with $T' < T$ and a probability measure P_τ^ε, such that

$$dX_t = [b_t(X)\,dt + a_t(X)\,1_{(t\,\geq\,\tau)}]\,dt + \varepsilon\,dW_t^{\varepsilon,\tau}\;, \quad X_0 = x_0\;, \quad 0 \leq t \leq T\;, \tag{2}$$

  where $\{W_t^{\varepsilon,\tau}\,,\,0 \leq t \leq T\}$ is a m–dimensional Wiener process under P_τ^ε.

Remark 2.1 The reason for requiring $T' < T$ is intuitively clear. It is impossible to detect a change occuring immediately before the end of the observation interval $[0,T]$, unless increasing dramatically the probabilty of false alarm. From the mathematical point of view, this will be reflected in the *detectability* assumptions (7), (13) and (22).

Remark 2.2 In the model presented above, it is assumed for simplicity that the diffusion coefficient is constant. However, all the results in this paper would generalize immediately to the case of a non–degenerate state dependent diffusion coefficient.

Let $\mathbf{L}$ denote the set of adapted non–anticipative functionals Φ defined on Ω, which are bounded and satisfy for some $C > 0$, the following Lipschitz condition

$$\sup_{0 \leq s \leq t} |\Phi_s(x) - \Phi_s(y)| \leq C \sup_{0 \leq s \leq t} |x_s - y_s|\;,$$

for all $0 \leq t \leq T$.

In the model above and throughout the paper, the coefficients are assumed to belong to $\mathbf{L}$, which implies existence and uniqueness of a solution to the stochastic differential equations of diffusion type (1) and (2), see Liptser–Shiryayev [4].

Introduce the following limiting deterministic systems, obtained in the limit when $\varepsilon \downarrow 0$, under the probability measure P^ε

$$\dot{x}_t = b_t(x)\;, \quad 0 \leq t \leq T\;,$$

and under the probability measure P_τ^ε

$$\dot{x}_t^\tau = b_t(x^\tau) + a_t(x^\tau)\,1_{(t\,\geq\,\tau)}\;, \quad 0 \leq t \leq T\;,$$

respectively. Under the Lipschitz assumption on the coefficients, it holds

$$\sup_{0 \le t \le T} |X_t - x_t| \le C\varepsilon \sup_{0 \le t \le T} |W_t^\varepsilon| \;,$$

$$\sup_{0 \le t \le T} |X_t - x_t^\tau| \le C\varepsilon \sup_{0 \le t \le T} |W_t^{\varepsilon,\tau}| \;,$$

so that the following exponential estimates hold for probabilities of large deviations

$$P^\varepsilon(\sup_{0 \le t \le T} |X_t - x_t| > \delta) \le C \;\exp\Big\{ -\frac{\delta^2}{KT\varepsilon^2} \Big\} \;,$$

$$P_\tau^\varepsilon(\sup_{0 \le t \le T} |X_t - x_t^\tau| > \delta) \le C \;\exp\Big\{ -\frac{\delta^2}{KT\varepsilon^2} \Big\} \;.$$

Let $\ell_\varepsilon(\tau)$ denote the suitably normalized log–likelihood function for estimating the change time τ

$$\ell_\varepsilon(\tau) = \varepsilon^2 \log \frac{dP_\tau^\varepsilon}{dP^\varepsilon} \;.$$

It holds

$$\ell_\varepsilon(\tau) = \varepsilon \int_\tau^T a_s(X)\, dW_s^\varepsilon - \tfrac{1}{2} \int_\tau^T |a_s(X)|^2\, ds \;.$$

Note that the expression used to compute the log–likelihood function is rather

$$\ell_\varepsilon(\tau) = \int_\tau^T a_s(X)\; [dX_s - b_s(X)\, ds] - \tfrac{1}{2} \int_\tau^T |a_s(X)|^2\, ds \;,$$

which depends only on the available observations $\{X_t \,,\; 0 \le t \le T\}$.

The alternate hypothesis (H_1) is a composite hypothesis, and the associated generalized likelihood ratio test (GLRT) for deciding between (H_0) and (H_1) is defined by the following region for rejecting (H_0)

$$D_\varepsilon = \Big\{ \sup_{0 \le \tau \le T'} \ell_\varepsilon(\tau) > c \Big\} \;,$$

where c is a given threshold. In other words, the null hypothesis (H_0) is rejected when $\omega \in D_\varepsilon$. In this case, the MLE for the change time τ is defined by

$$\widehat{\tau}_\varepsilon \in \operatorname*{argmax}_{0 \le \tau \le T'} \; \ell_\varepsilon(\tau) \;.$$

Under the alternate hypothesis (H_1), the behaviour (consistency, asymptotic probability distribution) of the estimate $\widehat{\tau}_\varepsilon$ has been investigated in [3] in the small noise asymptotics $\varepsilon \downarrow 0$. The purpose of this paper is to prove that both the probability of false alarm and the probability of miss detection associated with the above GLRT, go to zero with exponential rate when $\varepsilon \downarrow 0$. These probabilities are defined respectively as

$$F_\varepsilon = P^\varepsilon(D_\varepsilon) \qquad \text{and} \qquad M_\varepsilon = \sup_{0 \le \tau_0 \le T'} P_{\tau_0}^\varepsilon(D_\varepsilon^c) \;.$$

Here and throughout the paper, τ_0 will denote the *true* change time, under the alternate hypothesis (H_1).

2.1 Probability of false alarm

The following expression holds for the normalized log–likelihood function

$$\ell_\varepsilon(\tau) = \varepsilon \int_\tau^T a_s(X)\, dW_s^\varepsilon - \tfrac{1}{2} \int_\tau^T |a_s(X)|^2\, ds\ ,$$

where $\{W_t^\varepsilon\ ,\ 0 \leq t \leq T\}$ is a m–dimensional Wiener process under the probability measure P^ε. Introduce the limiting expression

$$\ell_0(\tau) = -\tfrac{1}{2} \int_\tau^T |a_s(x)|^2\, ds\ ,$$

where $\{x_t\ ,\ 0 \leq t \leq T\}$ is the solution of the limiting deterministic system

$$\dot{x}_t = b_t(x)\ , \quad 0 \leq t \leq T\ .$$

It is easily checked that the mapping $\tau \mapsto \ell_0(\tau)$ achieves its maximum for $\tau = T'$, i.e.

$$\ell_0^* \stackrel{\triangle}{=} \sup_{0 \leq \tau \leq T'} \ell_0(\tau) = \ell_0(T') = -\tfrac{1}{2} \int_{T'}^T |a_s(x)|^2\, ds \leq 0\ . \tag{3}$$

Proposition 2.3 *The probability of false alarm F_ε satisfies*

$$F_\varepsilon \leq C\ \exp\Big\{ -\frac{\delta^2}{KT\varepsilon^2} \Big\}\ ,$$

provided the threshold c satisfies

$$\delta = c - \ell_0^* > 0\ .$$

PROOF. Assume the following large deviation estimate

$$P^\varepsilon(\sup_{0 \leq \tau \leq T'} |\ell_\varepsilon(\tau) - \ell_0(\tau)| > \delta) \leq C\ \exp\Big\{ -\frac{\delta^2}{KT\varepsilon^2} \Big\}\ , \tag{4}$$

holds. Then

$$\ell_\varepsilon(\tau)\ \leq\ \ell_0(\tau) + |\ell_\varepsilon(\tau) - \ell_0(\tau)|$$

$$\sup_{0 \leq \tau \leq T'} \ell_\varepsilon(\tau)\ \leq\ \ell_0^* + \sup_{0 \leq \tau \leq T'} |\ell_\varepsilon(\tau) - \ell_0(\tau)|$$

implies

$$F_\varepsilon = P^\varepsilon(\sup_{0 \leq \tau \leq T'} \ell_\varepsilon(\tau) > c) \leq P^\varepsilon(\sup_{0 \leq \tau \leq T'} |\ell_\varepsilon(\tau) - \ell_0(\tau)| > \delta) \leq C\ \exp\Big\{ -\frac{\delta^2}{KT\varepsilon^2} \Big\}\ .$$

The remaining of the proof is devoted to proving estimate (4). Obviously

$$\ell_\varepsilon(\tau) - \ell_0(\tau) = \varepsilon \int_\tau^T a_s(X)\, dW_s^\varepsilon - \tfrac{1}{2} \int_\tau^T \Big\{ |a_s(X)|^2 - |a_s(x)|^2 \Big\}\, ds = I' + I''\ .$$

$\square$ *Study of I' :*

$$I' = \varepsilon \int_0^T a_s(X)\, dW_s^\varepsilon - \varepsilon \int_0^\tau a_s(X)\, dW_s^\varepsilon \ ,$$

therefore

$$\sup_{0 \leq \tau \leq T'} |I'| \leq \varepsilon \ | \int_0^T a_s(X)\, dW_s^\varepsilon | + \varepsilon \sup_{0 \leq \tau \leq T'} | \int_0^\tau a_s(X)\, dW_s^\varepsilon | \ ,$$

and standard exponential bound for stochastic integrals gives

$$P^\varepsilon(\sup_{0 \leq \tau \leq T'} |I'| > \tfrac{1}{2}\delta) \leq C \ \exp\Big\{ - \frac{\delta^2}{KT\varepsilon^2} \Big\} \ .$$

$\square$ *Study of I'' :*

$$I'' = -\tfrac{1}{2} \int_\tau^T [a_s(X) - a_s(x)]\ [a_s(X) + a_s(x)]\, ds \ ,$$

so that

$$\sup_{0 \leq \tau \leq T'} |I''| \leq C \sup_{0 \leq t \leq T} |X_t - x_t| \leq C\varepsilon \sup_{0 \leq t \leq T} |W_t^\varepsilon| \ .$$

Therefore

$$P^\varepsilon(\sup_{0 \leq t \leq T'} |I''| > \tfrac{1}{2}\delta) \leq C \ \exp\Big\{ - \frac{\delta^2}{KT\varepsilon^2} \Big\} \ . \qquad \square$$

2.2 Probability of miss detection

For the model given by equations (1) and (2)

$$\varepsilon\, dW_t^\varepsilon = a_t(X)\ 1_{(t \,\geq\, \tau_0)}\, dt + \varepsilon\, dW_t^{\varepsilon,\tau_0} \ , \quad 0 \leq t \leq T \ ,$$

and the following expression holds for the normalized log–likelihood function

$$\begin{aligned}
\ell_\varepsilon(\tau) \ &= \ \varepsilon \int_\tau^T a_s(X)\, dW_s^\varepsilon - \tfrac{1}{2} \int_\tau^T |a_s(X)|^2\, ds \\
&= \int_\tau^T a_s(X)\ [a_s(X)\ 1_{(s \,\geq\, \tau_0)}\, ds + \varepsilon\, dW_s^{\varepsilon,\tau_0}] - \tfrac{1}{2} \int_\tau^T |a_s(X)|^2\, ds \\
&= \varepsilon \int_\tau^T a_s(X)\, dW_s^{\varepsilon,\tau_0} + \int_{\tau \vee \tau_0}^T |a_s(X)|^2\, ds - \tfrac{1}{2} \int_\tau^T |a_s(X)|^2\, ds \\
&= \varepsilon \int_\tau^T a_s(X)\, dW_s^{\varepsilon,\tau_0} - \tfrac{1}{2} \int_\tau^{\tau \vee \tau_0} |a_s(X)|^2\, ds + \tfrac{1}{2} \int_{\tau \vee \tau_0}^T |a_s(X)|^2\, ds \ ,
\end{aligned}$$

where $\{W_t^{\varepsilon,\tau_0}\ ,\ 0 \leq t \leq T\}$ is a m–dimensional Wiener process under the probability measure $P^\varepsilon_{\tau_0}$. Introduce the limiting expression

$$\ell(\tau_0,\tau) = -\tfrac{1}{2}\int_\tau^{\tau\vee\tau_0} |a_s(x^{\tau_0})|^2\, ds + \tfrac{1}{2}\int_{\tau\vee\tau_0}^T |a_s(x^{\tau_0})|^2\, ds \ ,$$

where for all $0 \leq \tau_0 \leq T'$, $\{x_t^{\tau_0}\ ,\ 0 \leq t \leq T\}$ is the solution of the limiting deterministic system

$$\dot{x}_t^{\tau_0} = b_t(x^{\tau_0}) + a_t(x^{\tau_0})\ 1_{(t\,\geq\,\tau_0)}\ , \quad 0 \leq t \leq T\ .$$

It is easily checked that the mapping $\tau \mapsto \ell(\tau_0,\tau)$ achieves its maximum for $\tau = \tau_0$, i.e.

$$\ell^*(\tau_0) \stackrel{\triangle}{=} \sup_{0\leq\tau\leq T'} \ell(\tau_0,\tau) = \ell(\tau_0,\tau_0) = +\tfrac{1}{2}\int_{\tau_0}^T |a_s(x^{\tau_0})|^2\, ds \geq 0\ , \tag{5}$$

and

$$\tau_0 \in M(\tau_0) \stackrel{\triangle}{=} \operatorname*{argmax}_{0\leq\tau\leq T'}\ \ell(\tau_0,\tau)\ .$$

Proposition 2.4 *The probability of miss detection M_ε satisfies*

$$M_\varepsilon \leq C\ \exp\Big\{ - \frac{\delta^2}{KT\varepsilon^2} \Big\}\ ,$$

provided the threshold c satisfies

$$\delta = \inf_{0\leq\tau_0\leq T'} \ \ell^*(\tau_0) - c > 0\ .$$

Proof. Assume the following large deviation estimate

$$P^\varepsilon_{\tau_0}(\sup_{0\leq\tau\leq T'} |\ell_\varepsilon(\tau) - \ell(\tau_0,\tau)| > \delta(\tau_0)) \leq C\ \exp\Big\{ - \frac{\delta^2}{KT\varepsilon^2} \Big\}\ , \tag{6}$$

holds. Then

$$\ell(\tau_0,\tau)\ \leq\ \ell_\varepsilon(\tau) + |\ell_\varepsilon(\tau) - \ell(\tau_0,\tau)|$$

$$\ell^*(\tau_0)\ \leq\ \sup_{0\leq\tau\leq T'} \ell_\varepsilon(\tau) + \sup_{0\leq\tau\leq T'} |\ell_\varepsilon(\tau) - \ell(\tau_0,\tau)|$$

implies

$$P^\varepsilon_{\tau_0}(\sup_{0\leq\tau\leq T'} \ell_\varepsilon(\tau) < c) \leq P^\varepsilon_{\tau_0}(\sup_{0\leq\tau\leq T'} |\ell_\varepsilon(\tau) - \ell(\tau_0,\tau)| > \delta(\tau_0)) \leq C\ \exp\Big\{ - \frac{\delta^2(\tau_0)}{KT\varepsilon^2} \Big\}\ ,$$

provided the threshold c satisfies

$$\delta(\tau_0) = \ell^*(\tau_0) - c > 0\ .$$

Moreover

$$M_\varepsilon = \sup_{0\le\tau_0\le T'} P^\varepsilon_{\tau_0}(\sup_{0\le\tau\le T'} \ell_\varepsilon(\tau) < c) \le C \ \exp\Big\{ -\frac{\delta^2}{KT\varepsilon^2} \Big\} \ ,$$

provided the threshold c satisfies

$$\delta = \inf_{0\le\tau_0\le T'} \delta(\tau_0) = \inf_{0\le\tau_0\le T'} \ \ell^*(\tau_0) - c > 0 \ .$$

The remaining of the proof is devoted to proving estimate (6). Obviously

$$\ell_\varepsilon(\tau) - \ell(\tau_0,\tau) \ = \ \varepsilon \int_\tau^T a_s(X)\, dW^{\varepsilon,\tau_0}_s - \tfrac{1}{2}\int_\tau^{\tau\vee\tau_0} \Big\{ |a_s(X)|^2 - |a_s(x^{\tau_0})|^2 \Big\}\, ds$$

$$+ \tfrac{1}{2}\int_{\tau\vee\tau_0}^T \Big\{ |a_s(X)|^2 - |a_s(x^{\tau_0})|^2 \Big\}\, ds = I' + I'' \ .$$

□ *Study of I' :*

$$I' = \varepsilon \int_0^T a_s(X)\, dW^{\varepsilon,\tau_0}_s - \varepsilon \int_0^\tau a_s(X)\, dW^{\varepsilon,\tau_0}_s \ ,$$

therefore

$$\sup_{0\le\tau\le T'} |I'| \le \varepsilon \ |\int_0^T a_s(X)\, dW^{\varepsilon,\tau_0}_s| + \varepsilon \sup_{0\le\tau\le T'} |\int_0^\tau a_s(X)\, dW^{\varepsilon,\tau_0}_s| \ ,$$

and standard exponential bound for stochastic integrals gives

$$P^\varepsilon_{\tau_0}(\sup_{0\le\tau\le T'} |I'| > \tfrac{1}{2}\delta) \le C \ \exp\Big\{ -\frac{\delta^2}{KT\varepsilon^2} \Big\} \ .$$

□ *Study of I'' :*

$$I'' \ = \ -\tfrac{1}{2}\int_\tau^{\tau\vee\tau_0} [a_s(X) - a_s(x^{\tau_0})] \ [a_s(X) + a_s(x^{\tau_0})]\, ds$$

$$+ \tfrac{1}{2}\int_{\tau\vee\tau_0}^T [a_s(X) - a_s(x^{\tau_0})] \ [a_s(X) + a_s(x^{\tau_0})]\, ds \ ,$$

so that

$$\sup_{0\le\tau\le T'} |I''| \le C \sup_{0\le t\le T} |X_t - x^{\tau_0}_t| \le C\varepsilon \sup_{0\le t\le T} |W^{\varepsilon,\tau_0}_t| \ .$$

Therefore

$$P^\varepsilon_{\tau_0}(\sup_{0\le t\le T'} |I''| > \tfrac{1}{2}\delta) \le C \ \exp\Big\{ -\frac{\delta^2}{KT\varepsilon^2} \Big\} \ . \qquad \square$$

2.3 Conclusion

It has been proved that both the probability of false alarm and the probability of miss detection go to zero with exponential rate when $\varepsilon \downarrow 0$, provided the threshold c satisfies simultaneously

$$\ell_0^* < c < \inf_{0 \le \tau_0 \le T'} \ \ell^*(\tau_0) \ ,$$

where ℓ_0^* and $\ell^*(\tau_0)$ are defined in (3) and (5) respectively. This is possible under the following *detectability* assumption

$$- \tfrac{1}{2} \int_{T'}^{T} |a_s(x)|^2 \, ds < \inf_{0 \le \tau_0 \le T'} \ \tfrac{1}{2} \int_{\tau_0}^{T} |a_s(x^{\tau_0})|^2 \, ds \ . \tag{7}$$

Note that for this assumption to hold, it is necessary that $T' < T$.

Considering the change time estimation, it follows from the large deviations estimate (6), that the following *consistency* result holds

$$P^\varepsilon_{\tau_0}(d(\widehat{\tau}^\varepsilon, M(\tau_0)) > \delta) \xrightarrow{\varepsilon \downarrow 0} 0 \ ,$$

where $\{\widehat{\tau}^\varepsilon \, , \, \varepsilon > 0\}$ is any MLE sequence for the change time, and

$$M(\tau_0) \stackrel{\triangle}{=} \operatorname*{argmax}_{0 \le \tau \le T'} \ \ell(\tau_0, \tau) \ ,$$

is the set–valued deterministic change time estimate. Under the following *identifiablity* assumption

$$|a_{\tau_0}(x^{\tau_0})| > 0 \ , \tag{8}$$

there is no other point than τ_0 in the set $M(\tau_0)$.

3 Robustness w.r.t. misspecificied change

In this section, the problem is again to decide whether a change has occured or not, based on the model described by equations (1) and (2), so that the normalized log–likelihood function is still given by

$$\ell_\varepsilon(\tau) = \varepsilon \int_\tau^T a_s(X)\, dW_s^\varepsilon - \tfrac{1}{2} \int_\tau^T |a_s(X)|^2\, ds \ ,$$

and the GLRT for deciding between (H_0) and (H_1) is still defined by the following region for rejecting (H_0)

$$D_\varepsilon = \Big\{ \sup_{0 \leq \tau \leq T'} \ell_\varepsilon(\tau) > c \Big\} \ ,$$

where c is a given threshold.

However, it is assumed that if a change actually occurs at time τ_0, then the *true* probability measure is $\bar{P}^\varepsilon_{\tau_0}$, such that

$$dX_t = [b_t(X) + \bar{a}_t(X)\, 1_{(t \geq \tau_0)}]\, dt + \varepsilon\, d\bar{W}_t^{\varepsilon,\tau_0} \ , \quad X_0 = x_0 \ , \quad 0 \leq t \leq T \ , \tag{9}$$

where $\{\bar{W}_t^{\varepsilon,\tau_0} \,,\, 0 \leq t \leq T\}$ is a m–dimensional Wiener process under $\bar{P}^\varepsilon_{\tau_0}$. Therefore, the probability of false alarm and the probability of miss detection associated with the above GLRT, are defined respectively as

$$F_\varepsilon = P^\varepsilon(D_\varepsilon) \qquad \text{and} \qquad \bar{M}_\varepsilon = \sup_{0 \leq \tau_0 \leq T'} \bar{P}^\varepsilon_{\tau_0}(D_\varepsilon^c) \ .$$

3.1 Probability of false alarm

Because there is no misspecification when no change occurs, the result of Proposition 2.3 still holds.

3.2 Probability of miss detection

For the model given by equations (1) and (9)

$$\varepsilon\, dW_t^\varepsilon = \bar{a}_t(X)\, 1_{(t \geq \tau_0)}\, dt + \varepsilon\, d\bar{W}_t^{\varepsilon,\tau_0} \ , \quad 0 \leq t \leq T \ ,$$

and the following expression holds for the normalized log–likelihood function

$$\begin{aligned} \ell_\varepsilon(\tau) &= \varepsilon \int_\tau^T a_s(X)\, dW_s^\varepsilon - \tfrac{1}{2} \int_\tau^T |a_s(X)|^2\, ds \\ &= \int_\tau^T a_s(X)\, [\bar{a}_s(X)\, 1_{(s \geq \tau_0)}\, ds + \varepsilon\, d\bar{W}_s^{\varepsilon,\tau_0}] - \tfrac{1}{2} \int_\tau^T |a_s(X)|^2\, ds \end{aligned}$$

$$= \varepsilon \int_\tau^T a_s(X)\, d\bar{W}_s^{\varepsilon,\tau_0} + \int_{\tau\vee\tau_0}^T a_s(X)\, \bar{a}_s(X)\, ds - \tfrac{1}{2} \int_\tau^T |a_s(X)|^2\, ds$$

$$= \varepsilon \int_\tau^T a_s(X)\, d\bar{W}_s^{\varepsilon,\tau_0} - \tfrac{1}{2} \int_\tau^{\tau\vee\tau_0} |a_s(X)|^2\, ds + \tfrac{1}{2} \int_{\tau\vee\tau_0}^T |\bar{a}_s(X)|^2\, ds$$

$$- \tfrac{1}{2} \int_{\tau\vee\tau_0}^T |a_s(X) - \bar{a}_s(X)|^2\, ds \ ,$$

where $\{\bar{W}_t^{\varepsilon,\tau_0}\, ,\, 0 \leq t \leq T\}$ is a m–dimensional Wiener process under the *true* probability measure $\bar{P}_{\tau_0}^\varepsilon$. Introduce the limiting expression

$$\ell(\tau_0,\tau) \;=\; -\tfrac{1}{2} \int_\tau^{\tau\vee\tau_0} |a_s(x^{\tau_0})|^2\, ds$$

$$+ \tfrac{1}{2} \int_{\tau\vee\tau_0}^T \Big\{ |\bar{a}_s(x^{\tau_0})|^2 - |a_s(x^{\tau_0}) - \bar{a}_s(x^{\tau_0})|^2 \Big\}\, ds \ ,$$

where for all $0 \leq \tau_0 \leq T'$, $\{x_t^{\tau_0}\, ,\, 0 \leq t \leq T\}$ is the solution of the limiting deterministic system

$$\dot{x}_t^{\tau_0} = b_t(x^{\tau_0}) + \bar{a}_t(x^{\tau_0})\; 1_{(t\,\geq\,\tau_0)}\ , \quad 0 \leq t \leq T \ .$$

Under the *consistency* assumption

$$\inf_{\tau_0 \leq t \leq T} \Big\{ |\bar{a}_t(x^{\tau_0})|^2 - |a_t(x^{\tau_0}) - \bar{a}_t(x^{\tau_0})|^2 \Big\} \geq 0 \ , \tag{10}$$

it is easily checked that the mapping $\tau \mapsto \ell(\tau_0,\tau)$ achieves its maximum for $\tau = \tau_0$, i.e.

$$\begin{aligned} \ell^*(\tau_0) &\stackrel{\triangle}{=} \sup_{0\leq\tau\leq T'} \ell(\tau_0,\tau) = \ell(\tau_0,\tau_0) \\ &= +\tfrac{1}{2} \int_{\tau_0}^T \Big\{ |\bar{a}_s(x^{\tau_0})|^2 - |a_s(x^{\tau_0}) - \bar{a}_s(x^{\tau_0})|^2 \Big\}\, ds \geq 0 \ , \end{aligned} \tag{11}$$

and

$$\tau_0 \in M(\tau_0) \stackrel{\triangle}{=} \operatorname*{argmax}_{0\leq\tau\leq T'}\ \ell(\tau_0,\tau) \ .$$

Note that the condition (10) is always satisfied if $a \equiv \bar{a}$. The condition (10) means roughly that along the *true* trajectory, the change to be detected should be larger than the misspecification error, see Figure 1. For example, in the case where $a_t(x) = \alpha$ and $\bar{a}_t(x) = \bar{\alpha}$, the *consistency* assumption (10) simply reduces to $|\bar{\alpha}| \geq |\alpha - \bar{\alpha}|$.

Proposition 3.1 *The probability of miss detection $\bar{M}_\varepsilon$ satisfies*

$$\bar{M}_\varepsilon \leq C\ \exp\Big\{ -\frac{\delta^2}{KT\varepsilon^2} \Big\} \ ,$$

provided a global consistency *assumption holds, i.e.*

$$\text{for all } 0 \leq \tau_0 \leq T', \text{ the consistency assumption (10) holds,} \tag{12}$$

and the threshold c satisfies

$$\delta = \inf_{0 \leq \tau_0 \leq T'} \ell^*(\tau_0) - c > 0 \ .$$

The proof of this result is similar to the proof of Proposition 2.4 above, and is therefore omitted.

3.3 Conclusion

It has been proved, under the *global consistency* assumption (12), that both the probability of false alarm and the probability of miss detection go to zero with exponential rate when $\varepsilon \downarrow 0$, provided the threshold c satisfies simultaneously

$$\ell_0^* < c < \inf_{0 \leq \tau_0 \leq T'} \ell^*(\tau_0) \ ,$$

where ℓ_0^* and $\ell^*(\tau_0)$ are defined in (3) and (11) respectively. This is possible under the following additional *detectability* assumption

$$-\tfrac{1}{2} \int_{T'}^{T} |a_s(x)|^2 \, ds < \inf_{0 \leq \tau_0 \leq T'} \tfrac{1}{2} \int_{\tau_0}^{T} \left\{ |\bar{a}_s(x^{\tau_0})|^2 - |a_s(x^{\tau_0}) - \bar{a}_s(x^{\tau_0})|^2 \right\} ds \ . \tag{13}$$

Note that for this assumption to hold, it is necessary that $T' < T$.

Considering the change time estimation, it follows from large deviations estimate similar to (6), that the following *consistency* result holds

$$\bar{P}^\varepsilon_{\tau_0}(d(\hat{\tau}^\varepsilon, M(\tau_0)) > \delta) \xrightarrow{\varepsilon \downarrow 0} 0 \ ,$$

where $\{\hat{\tau}^\varepsilon \, , \, \varepsilon > 0\}$ is any MLE sequence for the change time, and

$$M(\tau_0) \triangleq \operatorname*{argmax}_{0 \leq \tau \leq T'} \ \ell(\tau_0, \tau) \ ,$$

is the set–valued deterministic change time estimate. If in addition to the *consistency* assumption (10), the following *identifiablity* assumption

$$|a_{\tau_0}(x^{\tau_0})| > 0 \qquad \text{and} \qquad |\bar{a}_{\tau_0}(x^{\tau_0})| > |a_{\tau_0}(x^{\tau_0}) - \bar{a}_{\tau_0}(x^{\tau_0})| \ , \tag{14}$$

holds, then there is no other point than τ_0 in the set $M(\tau_0)$. Note that the condition (14) reduces to the condition (8) if $a \equiv \bar{a}$.

Therefore, it is possible both to detect a change and to estimate the change time, which is a *robustness* result, provided the change to be estimated is larger then the mis-specification error.

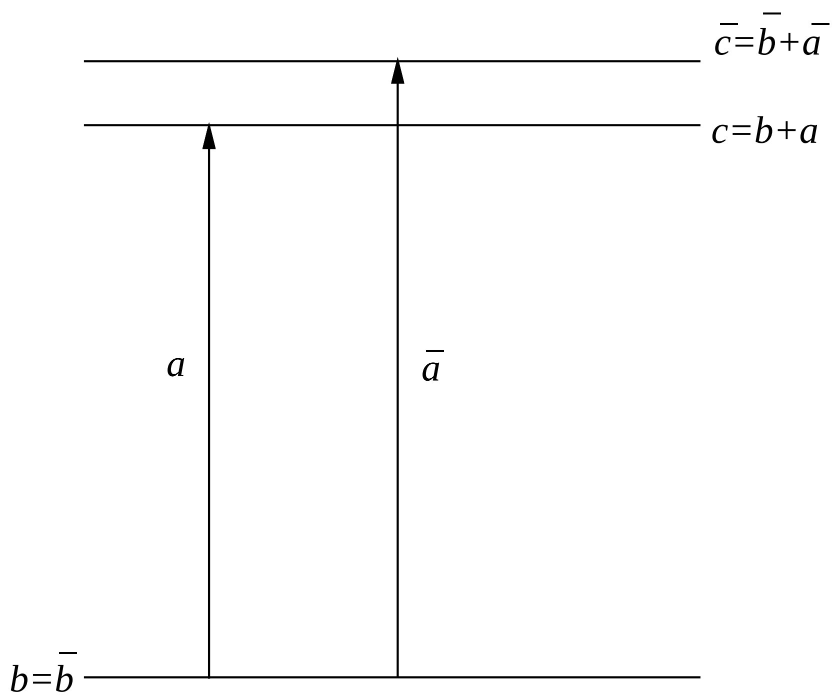

Figure 1: Consistency assumption for misspecified change under (H_1)

$$|\bar{a}| > |a - \bar{a}|$$

4 Robustness w.r.t. misspecified model

In this section, the problem is again to decide whether a change has occured or not, based on the model described by equations (1) and (2), so that the normalized log–likelihood function is still given by

$$\ell_\varepsilon(\tau) = \varepsilon \int_\tau^T a_s(X)\, dW_s^\varepsilon - \tfrac{1}{2} \int_\tau^T |a_s(X)|^2\, ds \ ,$$

and the GLRT for deciding between (H_0) and (H_1) is still defined by the following region for rejecting (H_0)

$$D_\varepsilon = \Big\{ \sup_{0 \le \tau \le T'} \ell_\varepsilon(\tau) > c \Big\} \ ,$$

where c is a given threshold.

However, it is assumed that if no change actually occurs, then the *true* probability measure is $\bar{P}^\varepsilon$, such that

$$dX_t = \bar{b}_t(X)\, dt + \varepsilon\, d\bar{W}_t^\varepsilon \ , \quad X_0 = x_0 \ , \quad 0 \le t \le T \ , \tag{15}$$

where $\{\bar{W}_t^\varepsilon \,, 0 \le t \le T\}$ is a m–dimensional Wiener process under $\bar{P}^\varepsilon$, and that in the alternate situation, if a change actually occurs at time τ_0, then the *true* probability measure is $\bar{P}_{\tau_0}^\varepsilon$, such that

$$dX_t = [\bar{b}_t(X) + \bar{a}_t(X)\, 1_{(t \,\ge\, \tau_0)}]\, dt + \varepsilon\, d\bar{W}_t^{\varepsilon,\tau_0} \ , \quad 0 \le t \le T \ , \tag{16}$$

where $\{\bar{W}_t^{\varepsilon,\tau_0} \,, 0 \le t \le T\}$ is a m–dimensional Wiener process under $\bar{P}_{\tau_0}^\varepsilon$.

As a conclusion, the probability of false alarm and the probability of miss detection associated with the above GLRT, are defined respectively as

$$\bar{F}_\varepsilon = \bar{P}^\varepsilon(D_\varepsilon) \qquad \text{and} \qquad \bar{M}_\varepsilon = \sup_{0 \le \tau_0 \le T'} \bar{P}_{\tau_0}^\varepsilon(D_\varepsilon^c) \ .$$

4.1 Probability of false alarm

For the model given by equations (1) and (15)

$$b_t(X)\, dt + \varepsilon\, dW_t^\varepsilon = \bar{b}_t(X)\, dt + \varepsilon\, d\bar{W}_t^\varepsilon \ , \quad 0 \le t \le T \ ,$$

and the following expression holds for the normalized log–likelihood function

$$\begin{aligned}
\ell_\varepsilon(\tau) &= \varepsilon \int_\tau^T a_s(X)\, dW_s^\varepsilon - \tfrac{1}{2} \int_\tau^T |a_s(X)|^2\, ds \\
&= \int_\tau^T a_s(X)\ \Big[[\bar{b}_s(X) - b_s(X)]\, ds + \varepsilon\, d\bar{W}_s^\varepsilon \Big] - \tfrac{1}{2} \int_\tau^T |a_s(X)|^2\, ds \\
&= \varepsilon \int_\tau^T a_s(X)\, d\bar{W}_s^\varepsilon + \int_\tau^T a_s(X)\ [\bar{b}_s(X) - b_s(X)]\, ds - \tfrac{1}{2} \int_\tau^T |a_s(X)|^2\, ds \ .
\end{aligned}$$

From the identity

$$a\,(\bar b - b) - \tfrac{1}{2}|a|^2 = -\tfrac{1}{2}|(b+a) - \bar b|^2 + \tfrac{1}{2}|b - \bar b|^2 \ ,$$

it follows that

$$\ell_\varepsilon(\tau) = \varepsilon \int_\tau^T a_s(X)\, d\bar W_s^\varepsilon - \tfrac{1}{2}\int_\tau^T |\, c_s(X) - \bar b_s(X)|^2\, ds + \tfrac{1}{2}\int_\tau^T |b_s(X) - \bar b_s(X)|^2\, ds \ ,$$

where $\{\bar W_t^\varepsilon\ , \ 0 \le t \le T\}$ is a m–dimensional Wiener process under the probability measure $\bar P^\varepsilon$, and $c \equiv b + a$ is the drift coefficient after the time change, for the *wrong* model. Introduce the limiting expression

$$\ell_0(\tau) = -\tfrac{1}{2}\int_\tau^T \Big\{ |\, c_s(x) - \bar b_s(x)|^2 - |b_s(x) - \bar b_s(x)|^2 \Big\}\, ds \ ,$$

where for all $0 \le \tau_0 \le T'$, $\{x_t\ , \ 0 \le t \le T\}$ is the solution of the limiting deterministic system

$$\dot x_t = \bar b_t(x) \ , \quad 0 \le t \le T \ .$$

Under the *consistency* assumption

$$\inf_{0 \le t \le T} \Big\{ |\, c_t(x) - \bar b_t(x)|^2 - |b_t(x) - \bar b_t(x)|^2 \Big\} \ge 0 \ , \tag{17}$$

it is easily checked that the mapping $\tau \mapsto \ell_0(\tau)$ achieves its maximum for $\tau = T'$, i.e.

$$\begin{aligned} \ell_0^* &\stackrel{\triangle}{=} \sup_{0 \le \tau \le T'} \ell_0(\tau) = \ell_0(T') \\ &= -\tfrac{1}{2}\int_{T'}^T \Big\{ |\, c_s(x) - \bar b_s(x)|^2 - |b_s(x) - \bar b_s(x)|^2 \Big\}\, ds \le 0 \ . \end{aligned} \tag{18}$$

Note that the condition (17) is always satisfied if $b \equiv \bar b$. The condition (17) means roughly that along the *true* trajectory, the change to be detected should be larger than the misspecification error, see Figure 2. For example, in the case where $b_t(x) = \beta^0$, $\bar b_t(x) = \bar\beta^0$ and $c_t(x) = \beta^1$, the assumption (17) simply reduces to $|\beta^1 - \bar\beta^0| \ge |\beta^0 - \bar\beta^0|$.

Proposition 4.1 *The probability of false alarm $\bar F_\varepsilon$ satisfies*

$$\bar F_\varepsilon \le C\ \exp\Big\{ -\frac{\delta^2}{KT\varepsilon^2} \Big\} \ ,$$

provided the consistency assumption (17) holds, and the threshold c satisfies

$$\delta = c - \ell_0^* > 0 \ .$$

The proof of this result is similar to the proof of Proposition 2.3 above, and is therefore omitted.

4.2 Probability of miss detection

For the model given by equations (1) and (16)

$$b_t(X)\,dt + \varepsilon\,dW^\varepsilon_t = [\bar{b}_t(X)\,dt + \bar{a}_t(X)\;1_{(t\,\geq\,\tau_0)}]\,dt + \varepsilon\,d\bar{W}^{\varepsilon,\tau_0}_t\;, \quad 0 \leq t \leq T\;,$$

and the following expression holds for the normalized log–likelihood function

$$\begin{aligned}
\ell_\varepsilon(\tau) \;&=\; \varepsilon \int_\tau^T a_s(X)\,dW^\varepsilon_s - \tfrac{1}{2}\int_\tau^T |a_s(X)|^2\,ds \\
&= \int_\tau^T a_s(X)\;\Big[[\bar{b}_s(X) - b_s(X)]\,ds + \bar{a}_s(X)\;1_{(t\,\geq\,\tau_0)}\,ds + \varepsilon\,d\bar{W}^{\varepsilon,\tau_0}_s\Big] \\
&\qquad - \tfrac{1}{2}\int_\tau^T |a_s(X)|^2\,ds \\
&= \varepsilon \int_\tau^T a_s(X)\,d\bar{W}^{\varepsilon,\tau_0}_s + \int_\tau^T a_s(X)\;[\bar{b}_s(X) - b_s(X)]\,ds + \int_{\tau\vee\tau_0}^T a_s(X)\;\bar{a}_s(X)\,ds \\
&\qquad - \tfrac{1}{2}\int_\tau^T |a_s(X)|^2\,ds\;.
\end{aligned}$$

From the identities

$$a\;(\bar{b} - b) - \tfrac{1}{2}|a|^2 = -\tfrac{1}{2}|(b+a) - \bar{b}|^2 + \tfrac{1}{2}|b - \bar{b}|^2\;,$$

and

$$a\;(\bar{b} - b) + a\;\bar{a} - \tfrac{1}{2}|a|^2 = \tfrac{1}{2}|b - (\bar{b} + \bar{a})|^2 - \tfrac{1}{2}|(b+a) - (\bar{b} + \bar{a})|^2\;,$$

it follows that

$$\begin{aligned}
\ell_\varepsilon(\tau) \;=\; \varepsilon \int_\tau^T a_s(X)\,d\bar{W}^{\varepsilon,\tau_0}_s \\
&- \tfrac{1}{2}\int_\tau^{\tau\vee\tau_0} |c_s(X) - \bar{b}_s(X)|^2\,ds + \tfrac{1}{2}\int_\tau^{\tau\vee\tau_0} |b_s(X) - \bar{b}_s(X)|^2\,ds \\
&+ \tfrac{1}{2}\int_{\tau\vee\tau_0}^T |b_s(X) - \bar{c}_s(X)|^2\,ds - \tfrac{1}{2}\int_{\tau\vee\tau_0}^T |c_s(X) - \bar{c}_s(X)|^2\,ds\;,
\end{aligned}$$

where $\{\bar{W}^{\varepsilon,\tau_0}_t\;,\;0 \leq t \leq T\}$ is a m–dimensional Wiener process under the *true* probability measure $\bar{P}^\varepsilon_{\tau_0}$, and $\bar{c} \equiv \bar{b} + \bar{a}$ (resp. $c \equiv b + a$) is the drift coefficient after the time change, for the *true* model (resp. for the *wrong* model). Introduce the limiting expression

$$\begin{aligned}
\ell(\tau_0, \tau) \;=\; -\tfrac{1}{2}\int_\tau^{\tau\vee\tau_0} \Big\{|c_s(x^{\tau_0}) - \bar{b}_s(x^{\tau_0})|^2 - |b_s(x^{\tau_0}) - \bar{b}_s(x^{\tau_0})|^2\Big\}\;ds \\
+ \tfrac{1}{2}\int_{\tau\vee\tau_0}^T \Big\{|b_s(x^{\tau_0}) - \bar{c}_s(x^{\tau_0})|^2 - |c_s(x^{\tau_0}) - \bar{c}_s(x^{\tau_0})|^2\Big\}\;ds\;,
\end{aligned}$$

where for all $0 \le \tau_0 \le T'$, $\{x_t^{\tau_0}\,,\, 0 \le t \le T\}$ is the solution of the limiting deterministic system

$$\dot{x}_t^{\tau_0} = \bar{b}_t(x^{\tau_0}) + \bar{a}_t(x^{\tau_0})\; 1_{(t \,\ge\, \tau_0)}\;, \quad 0 \le t \le T\;.$$

Under the *consistency* assumption

$$\begin{aligned} &\inf_{0 \le t \le \tau_0} \Big\{ |c_s(x^{\tau_0}) - \bar{b}_s(x^{\tau_0})|^2 - |b_s(x^{\tau_0}) - \bar{b}_s(x^{\tau_0})|^2 \Big\} \ge 0\;, \\ &\inf_{\tau_0 \le t \le T} \Big\{ |b_s(x^{\tau_0}) - \bar{c}_s(x^{\tau_0})|^2 - |c_s(x^{\tau_0}) - \bar{c}_s(x^{\tau_0})|^2 \Big\} \ge 0\;, \end{aligned} \tag{19}$$

it is easily checked that the mapping $\tau \mapsto \ell(\tau_0, \tau)$ achieves its maximum for $\tau = \tau_0$, i.e.

$$\begin{aligned} \ell^*(\tau_0) &\stackrel{\triangle}{=} \sup_{0 \le \tau \le T'} \ell(\tau_0, \tau) = \ell(\tau_0, \tau_0) \\ &= +\tfrac{1}{2} \int_{\tau_0}^{T} \Big\{ |b_s(x^{\tau_0}) - \bar{c}_s(x^{\tau_0})|^2 - |c_s(x^{\tau_0}) - \bar{c}_s(x^{\tau_0})|^2 \Big\}\, ds \ge 0\;, \end{aligned} \tag{20}$$

and

$$\tau_0 \in M(\tau_0) \stackrel{\triangle}{=} \operatorname*{argmax}_{0 \le \tau \le T'}\; \ell(\tau_0, \tau)\;.$$

Note that the condition (19) reduces to the condition (10) if $b \equiv \bar{b}$, and is always satisfied if in addition $a \equiv \bar{a}$. The condition (19) means roughly that along the *true* trajectory, the change to be detected should be larger than the misspecification error, see Figure 3. For example, in the case where $b_t(x) = \beta^0$, $\bar{b}_t(x) = \bar{\beta}^0$, $c_t(x) = \beta^1$ and $\bar{c}_t(x) = \bar{\beta}^1$, the *consistency* assumption (19) simply reduces to $|\beta^1 - \bar{\beta}^0| \ge |\beta^0 - \bar{\beta}^0|$ and $|\beta^0 - \bar{\beta}^1| \ge |\beta^1 - \bar{\beta}^1|$.

Proposition 4.2 *The probability of miss detection $\bar{M}_\varepsilon$ satisfies*

$$\bar{M}_\varepsilon \le C\; \exp\Big\{ -\frac{\delta^2}{KT\varepsilon^2} \Big\}\;,$$

provided a global consistency *assumption holds, i.e.*

$$\textit{for all } 0 \le \tau_0 \le T', \textit{ the } \text{consistency } \textit{assumption (19) holds,} \tag{21}$$

and the threshold c satisfies

$$\delta = \inf_{0 \le \tau_0 \le T'}\; \ell^*(\tau_0) - c > 0\;.$$

The proof of this Proposition is similar to the proof of Proposition 2.4 above, and is therefore omitted.

4.3 Conclusion

It has been proved, under the *global consistency* assumption (21), that both the probability of false alarm and the probability of miss detection go to zero with exponential rate when $\varepsilon \downarrow 0$, provided the threshold c satisfies simultaneously

$$\ell_0^* < c < \inf_{0 \leq \tau_0 \leq T'} \ell^*(\tau_0) \ ,$$

where ℓ_0^* and $\ell^*(\tau_0)$ are defined in (18) and (20) respectively. This is possible under the following additional *detectability* assumption

$$\begin{aligned} -\tfrac{1}{2} \int_{T'}^{T} & \left\{ |\, c_s(x) - \bar{b}_s(x)|^2 - |b_s(x) - \bar{b}_s(x)|^2 \right\} \, ds \\ & < \inf_{0 \leq \tau_0 \leq T'} \ \tfrac{1}{2} \int_{\tau_0}^{T} \left\{ |b_s(x^{\tau_0}) - \bar{c}_s(x^{\tau_0})|^2 - |c_s(x^{\tau_0}) - \bar{c}_s(x^{\tau_0})|^2 \right\} \, ds \ . \end{aligned} \tag{22}$$

Note that for this assumption to hold, it is necessary that $T' < T$.

Considering the change time estimation, it follows from large deviations estimate similar to (6), that the following *consistency* result holds

$$P^{\varepsilon}_{\tau_0}(d(\hat{\tau}^{\varepsilon}, M(\tau_0)) > \delta) \xrightarrow{\varepsilon \downarrow 0} 0 \ ,$$

where $\{\hat{\tau}^{\varepsilon} \, , \, \varepsilon > 0\}$ is any MLE sequence for the change time, and

$$M(\tau_0) \stackrel{\triangle}{=} \operatorname*{argmax}_{0 \leq \tau \leq T'} \ \ell(\tau_0, \tau) \ ,$$

is the set–valued deterministic change time estimate. If in addition to the *consistency* assumption (19), the following *identifiablity* assumption

$$\begin{aligned} |c_{\tau_0}(x^{\tau_0}) - \bar{b}_{\tau_0}(x^{\tau_0})| &> |b_{\tau_0}(x^{\tau_0}) - \bar{b}_{\tau_0}(x^{\tau_0})| \ , \\ |b_{\tau_0}(x^{\tau_0}) - \bar{c}_{\tau_0}(x^{\tau_0})| &> |c_{\tau_0}(x^{\tau_0}) - \bar{c}_{\tau_0}(x^{\tau_0})| \ , \end{aligned} \tag{23}$$

holds, then there is no other point than τ_0 in the set $M(\tau_0)$. Note that the condition (23) reduces to the condition (14) if $b \equiv \bar{b}$.

As a conclusion, it is possible both to detect a change and to estimate the change time, which is a *robustness* result, provided the change to be estimated is larger then the misspecification error.

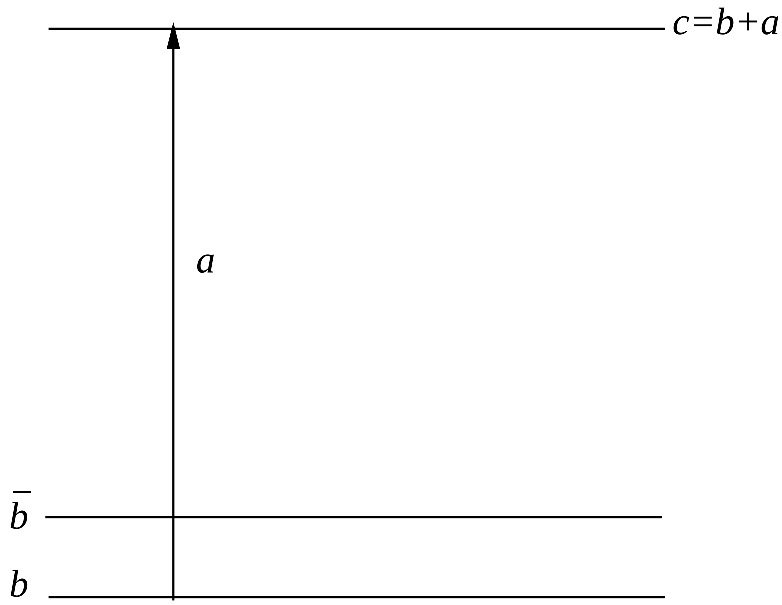

Figure 2: Consistency assumption for misspecified model under (H_0)

$$|c - \bar{b}| > |b - \bar{b}|$$

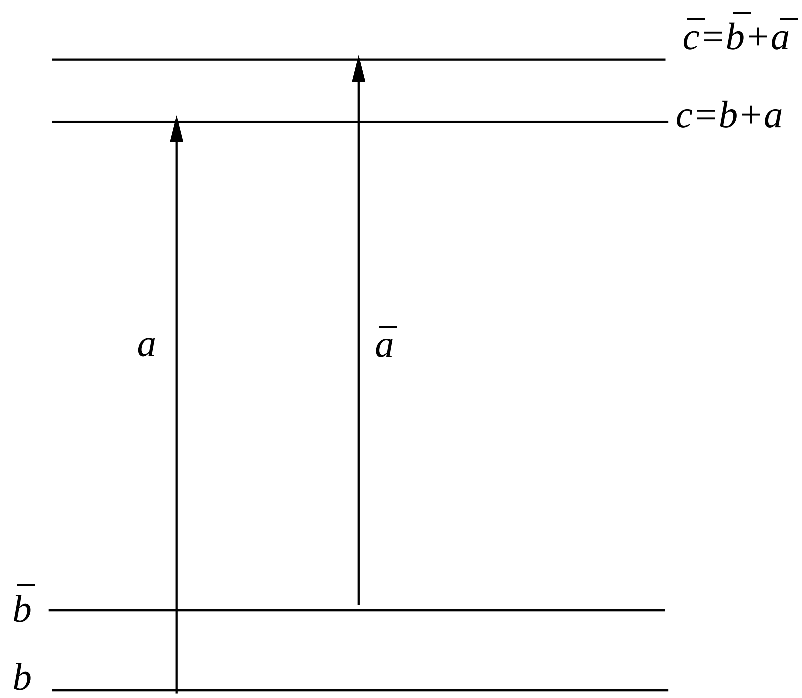

Figure 3: Consistency assumption for misspecified model under (H_1)

$|c-\bar{b}| > |b-\bar{b}|$ before change time

$|b-\bar{c}| > |c-\bar{c}|$ after change time

References

[1] M. BASSEVILLE and A. BENVENISTE, editors. *Detection of Abrupt Changes in Signals and Dynamical Systems*, volume 77 of *Lecture Notes in Control and Information Sciences*. Springer Verlag, 1986.

[2] I.A. IBRAGIMOV and R.Z. KHASMINSKII. *Statistical Estimation. Asymptotic Theory*, volume 16 of *Applications of Mathematics*. Springer Verlag, New York, 1981.

[3] Yu.A. KUTOYANTS. *Parameter Estimation for Stochastic Processes*, volume 6 of *Research and Exposition in Mathematics*. Heldermann Verlag, Berlin, 1984.

[4] R.Sh. LIPTSER and A.N. SHIRYAYEV. *Statistics of Random Processes I. General Theory*, volume 5 of *Applications of Mathematics*. Springer Verlag, New–York, 1977.

[5] I.W. McKEAGUE. Estimation for diffusion processes under misspecified models. *Journal of Applied Probability*, 21(3):511–520, 1984.

[6] A.N. SHIRYAYEV. *Optimal Stopping Rules*, volume 8 of *Applications of Mathematics*. Springer Verlag, New York, 1978.

[7] L.Yu. VOSTRIKOVA. Detection of a “disorder” in a Wiener process. *Theory of Probability and its Applications*, 26(2):356–362, 1981.